GROUP I. PART II.—*Continued.*

$\Delta =$

25.* $$\frac{1}{3}\frac{m_a^2 + m_b^2 + m_c^2}{\cot A + \cot B + \cot C}$$

26. $$\sqrt{2R\rho_a\rho_b\rho_c\left[1 + \cos(A - B) + \cos(B - C) + \cos(C - A)\right]}$$
$$= 2\sqrt{R\rho_a\rho_b\rho_c \sin(A + \tfrac{1}{2}B)\sin(B + \tfrac{1}{2}C)\sin(C + \tfrac{1}{2}A)}$$

27. $$R\sqrt[3]{h_a h_b h_c \sin A \sin B \sin C}$$

28. $$\tfrac{1}{3}R \sin A \sin B \sin C \left(\frac{ab}{h_c} + \frac{bc}{h_a} + \frac{ca}{h_b}\right)$$

29. $$\frac{h_a + h_b + h_c}{2\left(\dfrac{\cos\frac{1}{2}A}{\rho_a} + \dfrac{\cos\frac{1}{2}B}{\rho_b} + \dfrac{\cos\frac{1}{2}C}{\rho_c}\right)}$$

30. $$\frac{\rho_a \sin(C + \frac{1}{2}A) + \rho_b \sin(A + \frac{1}{2}B) + \rho_c \sin(B + \frac{1}{2}C)}{2\left(\dfrac{\cos\frac{1}{2}A}{\rho_a} + \dfrac{\cos\frac{1}{2}B}{\rho_b} + \dfrac{\cos\frac{1}{2}C}{\rho_c}\right)}$$

31. $$\frac{\rho_a\rho_b\rho_c\sqrt{(m_a^2 - h_a^2)(m_b^2 - h_b^2)(m_c^2 - h_c^2)}}{s(a - b)(b - c)(c - a)}.$$

[TO BE CONTINUED].

DEMONSTRATION OF DESCARTES'S THEOREM AND EULER'S THEOREM.

By PROF. G. B. HALSTED, Austin, Texas.

DESCARTES'S THEOREM.

Cutting by diagonals the faces not triangles into triangles, the whole surface of any polyhedron contains a number of triangular faces four less than double the number of summits.

Proof.

For, joining all the summits by a single closed broken line, this cuts the surface into two skew polygons, each of which contains $S - 2$ triangles, where S is the number of summits.

* *Tidsskrift for Mathematik*, 8° Copenhagen, 1883, fifth series, first year, No. 4, p. 136.

EULER'S THEOREM.

The number of faces and summits in any polyhedron taken together exceeds by two the number of its edges.

Proof.

First Case. If all the faces are triangles; then by Descartes's theorem,

$$F = 2(S - 2).$$

But also $2E = 3F$, since each edge belongs to two faces, and so we get a triangle for every time 3 is contained in $2E$. By adding, we have

$$2E = 2F + 2(S - 2),$$

that is,

$$F + S = E + 2.$$

Second Case. If not, all the faces are triangles. Since to any summit go as many faces as edges, we may replace any polygonal face by a pyramidal summit without changing the equality or inequality relation of $F + S$ to $E + 2$; for such replacement only adds the same number to F as to E and changes one face to a summit. But after all polygonal faces have been so replaced, $F + S = E + 2$ by our first case. Therefore the relation is always equality.

[These theorems are substantially one; the second is also due to Descartes, having been published in 1860 in his *Œuvres Inédites.—W. M. T.*]

PROOF OF A PROPOSITION IN MODERN GEOMETRY.

By MR. R. D. BOHANNAN, University of Virginia.

A,B are any two points on the curve which is the locus of the intersection of corresponding rays of two homographic ray-systems. If A,B be made the centres of two ray-systems whose corresponding rays intersect on this curve, these two ray-systems are homographic.

The curve which is the locus of corresponding rays of two homographic ray-systems is of the second degree and passes through the two ray-centres. Being of the second degree, it is fixed by fixing on it five points A,B,C,D,E. Take on it any sixth point F. The three rays BC, BD, BE may be taken arbitrarily to correspond to the three rays AC, AD, AE. Suppose the ray BK corresponding to BF does not intersect AF in F, but in K. Then we have two curves of the second degree, one passing through the six points A,B,C,D,E,F and the other passing through the six points A,B,C,D,E,K. But the curves have five points in common. Thus K,F are coincident. Therefore, etc.